UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics M252: Probability and Statistics

COURSE CODE	:	**MATHM252**
UNIT VALUE	:	**0.50**
DATE	:	**21-MAY-04**
TIME	:	**10.00**
TIME ALLOWED	:	**2 Hours**

04-C0964-3-130

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is *permitted in this examination.*

New Cambridge Statistical Tables are provided.

1. (a) State the conditions required for a function f to be the probability density function of a continuous random variable.

   (b) State the conditions required for a function F to be the distribution function of a continuous random variable.

   (c) A continuous random variable X has probability density function

$$f(x) = abx^{b-1}e^{-ax^b} \qquad (x > 0)$$

   for some parameters $a > 0$, $b > 0$. Show that the corresponding distribution function is

$$F(x) = 1 - e^{-ax^b} \qquad (x > 0) .$$

   Hence, or otherwise, show that $Y = X^b$ has an exponential distribution. What is the parameter of this distribution?

2. (a) Define the terms *event* and *random variable*, as used in probability theory. What is meant by saying that an event E *occurs*?

   (b) For any event E, let I_E denote a random variable taking the value 1 if E occurs, and 0 otherwise. Also, let p_E denote the probability of E. Write down the probability mass function of I_E; name its distribution, and write down its expected value.

   Let E^c denote the complement of E. Give an expression for I_{E^c} in terms of I_E.

   (c) Now consider two events A and B. Show that, using the same notational conventions as in part (b), the product $I_A I_B$ is equal to $I_{A\cap B}$. Hence write down the expected value of $I_A I_B$ in terms of $p_{A\cap B}$.

   (d) Show that $I_{(A\cup B)^c} = (1-I_A)(1-I_B)$. Hence find an expression for the expected value of $I_{A\cup B}$. Deduce that

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) .$$

3. (a) In a certain university, exam papers are set with the intention that students' marks will be normally distributed with a mean of 55% and a standard deviation of 10%. Assuming that this intended distribution is correct, what proportion of students would you expect to obtain marks:

   (i) under 35%?

   (ii) over 70%?

   (iii) between 50% and 70%?

   (b) Students take 4 exams in their second year. They must pass at least 3 of these exams in order to progress to their third year. The pass mark is 35%. Assuming that each student's marks on different exams are independent, with the intended distribution given in part (a), what proportion of second year students would you expect to progress to the third year?

   (c) The university is reviewing its examination arrangements. It is proposed that, instead of requiring passes in at least 3 exams, a student can progress to their third year if their average mark across all four exams is at least 45%.

   (i) State the distribution of a student's average mark, under the same assumptions as previously.

   (ii) Under the proposed scheme, what proportion of students would you expect to progress to their third year?

   (iii) It is also suggested that the format of the exam papers should be changed, so that that the marks on any individual paper are normally distributed with mean 55% and standard deviation σ%. What value of σ should be chosen to ensure that the same proportion of students progress to the third year under the new scheme as under the old?

4. (a) The random variable X has a Poisson distribution with mean μ. Show that the probability generating function (PGF) of X is given by $\Pi(s) = e^{(s-1)\mu}$, for all s.

   (b) Suppose now that X_1 and X_2 are independent Poisson random variables with means μ_1 and μ_2 respectively. Use probability generating functions to show that $X_1 + X_2$ also has a Poisson distribution, and state the parameter of this distribution. (You may use, without proof, the fact that the PGF of a sum of independent variables is equal to the product of their individual PGFs).

   By considering the Poisson Process, explain why this result is 'obvious'.

   (c) Let X_1 and X_2 be independent Poisson random variables as in part (b). It is observed that $X_1 + X_2 = 1$. Given this information, what is the probability that $X_1 = 1$?

5. A car hire company is trying to decide whether the use of a new brand (B) of tyres, rather than the conventional brand (A), affects fuel consumption. Twelve cars of the same model and age were equipped with brand B tyres and driven (by twelve different drivers) over a prescribed test course. Then the tyres on the cars were replaced by brand A tyres, and each driver drove the same car round the course again. The petrol consumption in kilometres per litre was recorded as follows:

Car	1	2	3	4	5	6	7	8	9	10	11	12
Brand A	4.1	4.9	6.2	6.9	6.8	4.4	5.7	5.8	6.9	4.7	6.0	4.9
Brand B	4.2	4.7	6.6	7.0	6.7	4.5	5.7	6.0	7.4	4.9	6.1	5.2

(a) Prepare a stem-and-leaf diagram showing the data for brand B.

(b) Carry out an appropriate statistical test, at the 5% level, to determine whether there is any evidence for a difference between the brands, in terms of fuel economy. State your conclusions clearly, along with any assumptions that underly your test procedure.

6. A battery manufacturer has two factories. At each factory, a proportion of the batteries produced are defective. As part of the manufacturer's quality control procedures, a sample of each day's output is examined and the number of defectives recorded. This is then used to estimate the true proportion of defective batteries being produced in each factory.

However, the sampling strategies at the two factories are different. At Factory 1 (where the true proportion of defectives is p_1, say), the quality control manager examines a batch of n batteries each day, and estimates p_1 as X/n, where X is the number of defectives in the batch. At Factory 2, where the true proportion of defectives is p_2, the manager examines batteries until r defectives have been found, and estimates p_2 as $(r-1)/(N-1)$, where N is the number of batteries tested up to and including the rth defective.

(a) Assuming that batteries are defective independently of each other at both factories, state the distributions of X and N, along with their expectations. Write down the probability mass function of N.

(b) Show that the quality control manager at Factory 1 is using an unbiased estimator. Calculate the standard error of this estimator, and explain (briefly) how this could be used to construct an approximate 95% confidence interval for p_1 if n is large.

(c) Show that, for $r > 1$, the quality control manager at Factory 2 is also using an unbiased estimator.

You may use, without proof, the identity $\sum_{j=m}^{\infty} \binom{j-1}{m-1} x^m (1-x)^{j-m} = 1$, for $0 < x < 1$.